AP* Calculus Review

Integration Techniques

Teacher Packet

Notes

When learning integration techniques, students will frequently comment, "I don't know what to use when." To help with this problem form a strategy list that has been prioritized - in order of what to try first. Start this list early and recycle it each time you add a new technique.
Note to teachers: Model the use of the system as you work problems in class. Students will generally try to use the newest technique on problems even if it is not the best one to use. Have problems in the practice/homework sets that use old methods so that students cannot focus just on the newest techniques.
Here is a possible strategy list.

1. Is there a theorem/formula that will apply to this integrand? Think in terms of derivatives that you have memorized.
2. In the case of a definite integral, can I use geometry to evaluate it quickly? Think in terms of a graph and a geometric shape.
3. Can I use algebra or a trig identity to rewrite the integrand? Think about doing implied operations like multiplying or dividing.
4. Can I see a u-sub format? Think about composite functions.
5. What about a change of variable? This technique is less frequently used that others previously mentioned.

Examples

1. $\int \frac{5}{x}\,dx$

This problem uses the theorem that allows you to move a constant outside the integral; it is then an application of an antiderivative formula.

$$\int \frac{5}{x}\,dx = 5\int \frac{1}{x}\,dx = 5\ln x + C$$

2. $\int_{-3}^{0} \sqrt{9-x^2}\,dx$

This problem does not fit any theorem or antiderivative formula; however, the graph of the function on the interval is a quarter circle in quadrant 2, centered at (0, 0), radius 3. Use the area formula to evaluate the integral.

$$\int_{-3}^{0} \sqrt{9-x^2}\,dx = \frac{1}{4}\pi(3)^2 = \frac{9\pi}{4}$$

3. $\int \tan^2 x\, dx$

This problem does not fit any formula/theorem, does not work with geometry, and there is no implied algebra. The trig identity, $1 + \tan^2 x = \sec^2 x$, will transform the integrand so that the integral can be broken into two parts.

$$\int \tan^2 x\, dx = \int (\sec^2 x - 1)\, dx = \int \sec^2 x\, dx - \int 1\, dx = \tan x - x + C$$

4. $\int x\cos(x^2)\, dx$

This problem does not fit any formula/theorem, does not work with geometry, and there is no implied algebra or trig identity that would help. However, you can see the composite function in the integrand – the function x^2 is "inside" the function $\cos x$. Try u-subs.

There are two methods for u-subs depending on the text that you use.

Method A

Let $u = x^2$, $du = 2x\, dx$. The factor of two is needed in the integrand; rewrite the integrand using a clever form of 1 (i.e., $\frac{2}{2}$). Using the theorem that allows you to move a constant outside the integral and write the antiderivative.

$$\int x\cos(x^2)\, dx = \int \frac{2}{2} x\cos(x^2)\, dx = \frac{1}{2}\int 2x\cos(x^2)\, dx = \frac{1}{2}\int \cos(x^2)\, 2x dx =$$

$$-\frac{1}{2}\sin(x^2) + C$$

Method B

Let $u = x^2$, $du = 2x\, dx$ or $x\, dx = \frac{1}{2}\, du$. Make the substitutions, apply the theorem that allows you to move the constant, and write the antiderivative. Go back to the original variable.

$$\int x\cos(x^2)\, dx = \int \cos u \frac{1}{2} du = \frac{1}{2}\int \cos u\, du = -\frac{1}{2}\sin(u) + C = -\frac{1}{2}\sin(x^2) + C$$

5. $\int x\sqrt{x-2}\,dx$ (assume $x > 2$)

As you go through the hierarchy of things to try you strike out at levels 1 – 4. But as you try u-subs, you see some potential. The process called change of variable is a "cousin" of u-subs – try it if u-sub isn't perfect for your problem. The process starts with choosing a u; however, you don't find the du at the second step. Instead, solve your "$u =$" equation for x and find dx. Then substitute throughout the entire integral, writing everything in terms of u - including rewriting the limits of integration, if needed. The next step is to find the antiderivative. The final step is to rewrite the answer in terms of the original variable, or in the case of a definite integral, evaluate using FTC. One way to recognize a problem that needs a change of variable is to note that the exponent outside a radical is <u>not</u> one less than the exponent under the radical.

$\int x\sqrt{x-2}\,dx$

Method A

Let $u = x-2$, then $x = u+2$ and $dx = du$.

$$\int x\sqrt{x-2}\,dx = \int (u+2)\sqrt{u}\,du = \int (u^{\frac{3}{2}} + 2u^{\frac{1}{2}})\,du = \frac{2}{5}u^{\frac{5}{2}} + \frac{4}{3}u^{\frac{3}{2}} + C$$

$$= \frac{2}{5}(x-2)^{\frac{5}{2}} + \frac{4}{3}(x-2)^{\frac{3}{2}} + C$$

Method B

Let $u = \sqrt{x-2}$, then $x = u^2 + 2$ and $dx = 2u\,du$.

$$\int x\sqrt{x-2}\,dx = \int (u^2+2)(u)(2udu) = 2\int (u^4 + 2u^2)\,du = 2(\frac{1}{5}u^5 + \frac{2}{3}u^3) + C$$

$$= \frac{2}{5}(x-2)^{\frac{5}{2}} + \frac{4}{3}(x-2)^{\frac{3}{2}} + C$$

In this method, if there had been limits of integration, the rewrite would include them.

$$\int_2^{11} x\sqrt{x-2}\,dx = 2\int_0^3 (u^4 + 2u^2)\,du$$

Challenging Situations

1. $\int \frac{x+2}{x^2+1}\,dx$ vs. $\int \frac{x^2+2}{x+1}\,dx$

These problems have a common look but involve different algebraic rewrites. The first problem needs to be separated into two integrals and the second one needs to have the division algorithm applied (quotient plus remainder over divisor). In the first problem there is the added challenge of two integrals that "look alike" but are worked very differently – one inverse trig and one natural logarithm.

$$\int \frac{x+2}{x^2+1}\,dx = \int \frac{x}{x^2+1}\,dx + 2\int \frac{1}{x^2+1}\,dx = \frac{1}{2}\ln|x^2+1| + 2\arctan x + C$$

$$\int \frac{x^2+2}{x+1}\,dx = \int\left(x-1+\frac{3}{x+1}\right)\,dx = \frac{1}{2}x^2 - x + 3\ln|x+1| + C$$

$$\begin{array}{r@{\,}l}
 & x-1 \\
x+1 \,\big)\!\!\! & \overline{x^2+0x+2} \\
 & \underline{x^2+x} \\
 & -x+2 \\
 & \underline{-x-1} \\
 & 3
\end{array}$$

2. $\int \frac{x}{1+x^4}\,dx$ vs. $\int \frac{x^3}{1+x^4}\,dx$

Again the student is faced with two integrals that have the same look (in the denominator) but cannot be worked the same way. The first one needs an algebraic rewrite and a u-sub: $1 + x^4 = 1 + (x^2)^2$ and let $u = x^2$. This one turns out to be an inverse tangent. The second is a u-sub with $u = 1 + x^4$ and integrates to a natural logarithm.

$$\int \frac{x}{1+x^4}\,dx = \frac{1}{2}\int \frac{2x}{1+(x^2)^2}\,dx = \frac{1}{2}\arctan\left(x^2\right) + C$$

$$\int \frac{x^3}{1+x^4}\,dx = \frac{1}{4}\int \frac{4x^3}{1+x^4}\,dx = \frac{1}{4}\ln\left|1+x^4\right| + C$$

It may help students to see problems like this in pairs so that they can compare and contrast the "look" of the problems.

PRACTICE PROBLEMS

All of these problems are intended to be worked without a calculator.

1. $\int \cos 3x \, dx =$

(A) $-3\sin 3x + C$ (B) $-\sin 3x + C$ (C) $-\frac{1}{3}\sin 3x + C$

(D) $\frac{1}{3}\sin 3x + C$ (E) $3\sin 3x + C$

2. $\int \frac{1-3y}{\sqrt{2y-3y^2}} \, dy =$

(A) $4\sqrt{2y-3y^2} + C$ (B) $2\sqrt{2y-3y^2} + C$

(C) $\frac{1}{2}\ln\left(\sqrt{2y-3y^2}\right) + C$ (D) $\frac{1}{4}\ln\left(\sqrt{2y-3y^2}\right) + C$

(E) $\sqrt{2y-3y^2} + C$

3. $\int \frac{dy}{\sqrt{1-4y^2}} =$

(A) $-\frac{1}{2}\sqrt{1-4y^2} + C$ (B) $\frac{1}{2}\sqrt{1-4y^2} + C$

(C) $\sin^{-1} 2y + C$ (D) $\frac{1}{2}\sin^{-1} 2y + C$

(E) $-\frac{1}{2}\sin^{-1} 2y + C$

4. $\int_0^{\frac{\pi}{4}} \frac{dx}{\cos^2 3x} =$

(A) -3 (B) -1 (C) $-\frac{1}{3}$ (D) $\frac{1}{3}$ (E) 3

5. $\int_e^{e^3} \frac{\ln x}{x}\,dx =$

(A) 2 (B) $\frac{5}{2}$ (C) 4 (D) $\frac{9}{2}$ (E) 8

6. $\int_1^4 |x-3|\,dx =$

(A) $-\frac{3}{2}$ (B) $\frac{3}{2}$ (C) $\frac{5}{2}$ (D) $\frac{9}{2}$ (E) 5

For 7 – 20, evaluate the integral. For each integral, assume a domain on which the integral is defined. Remember – no calculator!

7. $\int \frac{x+1}{x^2-1}\,dx$

8. $\int \frac{e^x}{1+e^x}\,dx$

9. $\int \frac{e^x}{1+e^{2x}}\,dx$

10. $\int \left(x-\frac{1}{2x}\right)^2\,dx$

11. $\int (x+2)\sqrt{x-3}\,dx$

12. $\int \sqrt{4-2x}\,dx$

13. $\int \frac{\cos(x-1)}{\sin^2(x-1)}\,dx$

14. $\int_0^9 e^{\ln\sqrt{x}}\, dx$

15. $\int_0^{\frac{\sqrt{2}}{2}} \frac{\arcsin x}{\sqrt{1-x^2}}\, dx$

16. $\int_0^{\frac{\pi}{12}} \tan 3x \sec^2 3x\, dx$

17. $\int_1^4 \frac{2^{\sqrt{x}}}{2\sqrt{x}}\, dx$

18. $\int_0^{\sqrt{\ln x}} xe^{x^2}\, dx$

19. $\int_0^{\frac{3}{4}} \frac{x}{\sqrt{1-x}}\, dx$

20. $\int_0^1 \frac{x}{\sqrt{8x^2+1}}\, dx$